ADAPTIVE SUPERRESOLUTION PHENOMENA IN ABERRANT ARRAYS

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Abstract– **Maximum-likelihood (ML) beamformers can perform accurate bearing estimation by taking advantage of the fact that for high input signal-to-noise ratios, the adaptive beamwidth is much narrower than the classical Rayleigh limit. This paper studies the effect of random antenna element errors on this "superresolution" phenomenon. The imperfections considered are amplitude and phase errors which are constant during the period of adaptation. With a single input point-source signal, it is shown that the ML beamformer half-power adaptive beamwidth is degraded by array aberrations, but it is still several times better than that of conventional nonadaptive beamformers, provided the input signal-to-noise ratio is 0 dB or higher. A simple scalar formula is presented for the ML half-power beamwidth in the presence of element errors for uniformly spaced line arrays, and simulations are described which verified its accuracy to within 4%. Lastly, two new on-line algorithms are proposed to make the superresolution concept robust to array imperfections.**

1. INTRODUCTION

Maximum-likelihood (ML) beamformers can perform accurate bearing estimation by taking advantage of the fact that for high input signal-to-noise ratios, the *adaptive beamwidth* is much narrower than the classical Rayleigh limit of λ/D radians, where λ is the radiation wavelength and D the aperture width. The objective of this paper is to study the effect of imperfectly known antenna elements on this *superresolution* phenomenon, by studying degradation in ML beamformer half-power adaptive beamwidth for a single source. This is the first step towards understanding another important problem, namely the effect of imperfectly known antenna elements on the ability to resolve closely spaced multiple sources. With a single input point-source signal, it is shown that the ML beamformer half-power adaptive beamwidth is degraded by array aberrations, but it is still several times better than that of conventional nonadaptive beamformers, provided the input signal-to-noise ratio is 0 dB or higher. This result suggests that superresolution is possible for the ML beamformer despite errors in the receiving elements.

The degradation in bearing estimation quality due to element imperfections is a result of both random pointing errors and an increase in half-power adaptive beamwidth (*HPBW*). A pointing error can exist when the value of *array gain* is not unity in the assumed *look-direction*. The pointing error is generally much less than *HPBW*, which means that most of the degradation in bearing estimation quality is due to increased *HPBW*, the subject of this paper.

All element imperfections considered here are assumed constant during the period of adaptation. The signal environment consists of one narrowband far-field signal, *no jammers*, and additive white receiver noise. The signal and noise are both zero-mean, wide-sense stationary, and statistically independent of each other. The propagation medium is *linear*, *homogeneous*, and *isotropic*. Use of the optimal Wiener weighting, which minimizes *mean-square error* (*MSE*), is assumed.

Capon *et al.* [1] in 1967 were apparently the first to recognize the ML phenomenon, defined as using a set of weights that minimizes the output power of an antenna subject to a simple unity gain constraint for look-direction signals.

Since that time, there has been much research on superresolution [2]. The result of Walach [3] is the one most relevant to this paper. He showed that for an equally spaced line array, it was possible to derive a simple scalar expression to predict *HPBW* of a converged narrowband ML beamformer.

With respect to the "imperfect array problem," several researchers in the past have noted that linearly constrained adaptive beamforming suffers from a *hypersensitivity* to array imperfections when the input signal-to-noise ratio (SNR_i) exceeds some threshold [4-5]. Here "noise" means receiver noise. The solution to the imperfect array problem most relevant to this paper is Zahm's [6] *artificial receiver noise injection* strategy (cf. §4.2).

In terms of implementation, ML beamformers can be built in several ways. One way is Frost's [7] beamformer, which minimizes output power subject to a constant gain constraint in an assumed look-direction. Another way is to use an Applebaum-Chapman-Griffiths-Jim [8] *generalized sidelobe canceller* (GSC), which realizes the constraints by preprocessing, and uses *unconstrained* adaptation.

This paper combines the work on the superresolution problem with the work on imperfect arrays in order to study the achievable *HPBW* of a ML beamformer in the presence of random element gain (amplitude and phase) errors. The analysis is based on use of the GSC. Contributions contained herein are:

- Extension of Walach's [3] result to imperfect arrays.
- A discussion of the effect of random element misplacement on *HPBW*.
- Two on-line algorithms for making *HPBW* of the ML beamformer robust to imperfections, both based on artificially injecting receiver noise into a ML GSC.

The outline of the paper is as follows: Section 2 presents the formula for *HPBW* in the presence of array imperfections. Section 3 describes simulations which verfied the accuracy of the *HPBW* formula to within 4%. Section 4 looks at degradation in *HPBW* due to imperfections in a more qualitative way, and also presents two new algorithms which make *HPBW* robust to imperfections. Section 5 considers the wideband case. Section 6 contains the conclusions.

2. THE MAIN RESULT

This section will present the formula from [2] for *HPBW* of the ML beamformer in the presence of random element gain imperfections.

The GSC is shown schematically in Fig. 1, consisting of K elements, which ideally would all be omnidirectional with identical amplitude and phase. A simple model for random antenna element imperfections is to let each element have a *random* complex gain g_i, assumed to remain *constant* during the period of adaptation. The use of a complex gain implicitly takes into account random element amplitude *and* phase errors. Denoting the zero-mean random element amplitude error at element i by Δa_i and the zero-mean phase error by Δp_i, the complex gain can be written as

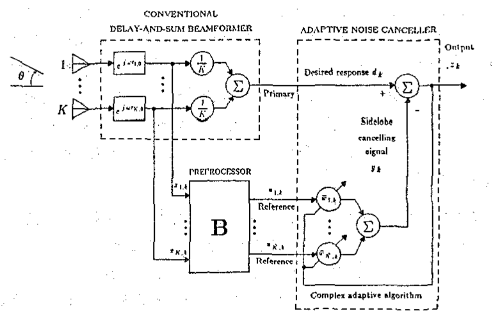

Fig. 1. Block diagram of narrowband GSC. The additive receiver noises following the steering delays are not shown.

$$g_i = (1 + \Delta a_i)\, e^{j \Delta p_i} = 1 + \Delta g_i \qquad (1)$$

where $j \triangleq \sqrt{-1}$, and Δg_i is the zero-mean complex gain error.

The adaptive beamformer with and without imperfections is illustrated in Fig. 2. Δp_i includes the phase error due to random element misplacement, which changes with signal direction [4], so therefore Δg_i and g_i change with signal direction. It is also conceivable that Δa_i could be a function of signal direction, for example if the antenna element pattern was not truly omnidirectional. In the presence of errors, the beamformer still adapts in such a way as to minimize *MSE*, but the fact that it is unaware of the errors corrupting the data causes a degradation in performance.

For the narrowband case, presteering the array to a known look-direction is accomplished by use of a phase shifter at the output of each antenna element. In order to steer the array to the look-direction θ_0, a presteering delay of $-\tau_{i,0}$ is needed at element i. In the absence of presteering, a signal coming from an angle θ_s would undergo a time delay of $\tau_{i,s}$. Thus, with look-direction steering, the signal undergoes a total time delay $\tau_i = \tau_{i,s} - \tau_{i,0}$ at element i.

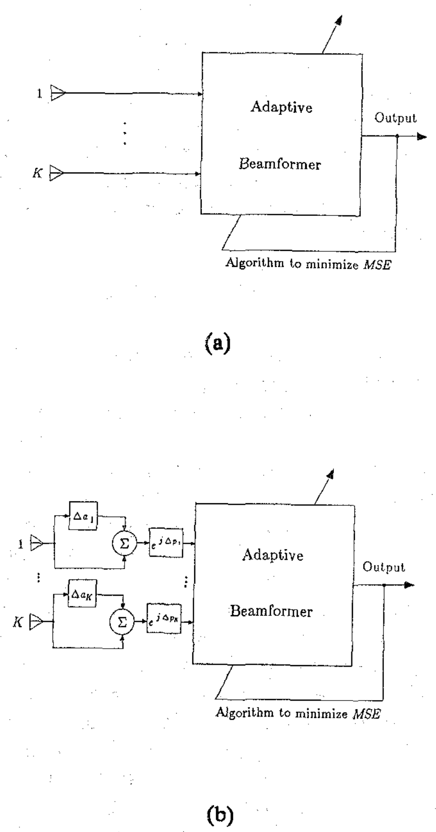

Fig. 2. Adaptive beamformer with and without imperfections. (a) without imperfections; (b) with direction-dependent imperfections.

Imperfections in the presteering electronics can also be included as part of the amplitude and phase error terms Δa_i and Δp_i.

After passing through the presteering delays, the signal received at each element is corrupted by additive zero-mean *white noise*, as shown in Fig. 3. This additive receiver noise is assumed to be *independent and identically distributed* (*i.i.d.*) from element to element.

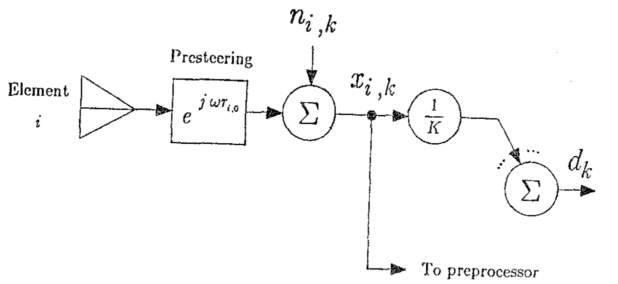

Fig. 3. Model of i-th receiver channel ($i=1, \ldots, K$).

The beamformer consists of two branches. The upper is termed the ***desired response*** branch, and its purpose is to form the desired response d_k, which is the ***primary*** input to the ***adaptive noise canceller***. In the absence of array imperfections, the desired response branch is constrained to have a unit look-direction gain. In general, this branch is a conventional ***delay-and-sum*** beamformer, with K nonadaptive weights fixed in such a way that the array beamwidth and average sidelobe level are both satisfactory [8]. This paper assumes ***uniform*** $1/K$ weighting, but other weightings could be easily considered.

The lower branch of the beamformer is the ***sidelobe cancelling*** branch. Its purpose is to form the sidelobe cancelling signal y_k by providing $\hat{K}$ ***reference*** inputs to the adaptive noise canceller. y_k contains estimates of non-look-direction components in the desired response, so that after subtracting y_k from d_k, the beamformer output z_k is a "cleaner" representation of look-direction components. Note the use of complex conjugate weights $\bar{w}_{1,k}, \cdots, \bar{w}_{K,k}$ in computing y_k. These weights can be updated by several different methods, for example the ***complex LMS*** algorithm [7].

The sidelobe cancelling branch is preceded by the ***signal blocking matrix*** $\mathbf{B}$, a preprocessor designed to block look-direction signals so that the sidelobe cancelling branch cannot learn them. The preprocessor has K inputs and $\hat{K}$ outputs. In this paper it is assumed that $\hat{K} < K$. The simplest example of a preprocessor is the use of adjacent element differencing, yielding zero gain in the look-direction (in the absence of imperfections). Use of the latter blocking processor makes the GSC behave like a ML Frost beamformer, as far as the Wiener solution is concerned [8]. With this type of preprocessor $\hat{K} = K-1$.

The restrictions on $\mathbf{B}$ are $\mathbf{B}\vec{1}_K = \vec{0}_{\hat{K}}$ and ***rank*** $(\mathbf{B}) = \hat{K}$, where $\vec{1}_K$ is a $K \times 1$ vector of 1's, and $\vec{0}_{\hat{K}}$ a $\hat{K} \times 1$ vector of 0's [8].

A class of signal blocking matrices known as ***central difference matrices*** is formed by using r cascaded columns of differencing, as shown in Fig. 4. Notationally, they are written $\mathbf{B}_K^{(r-1)}$, where the subscript indicates the number of antenna elements, and the superscript $(r-1)$ the use of a main beam zero $(r-1)$-st derivative constraint in the look-direction. The quantity $\hat{K}$ (the dimension of the state and weight vectors) then becomes $(K-r)$.

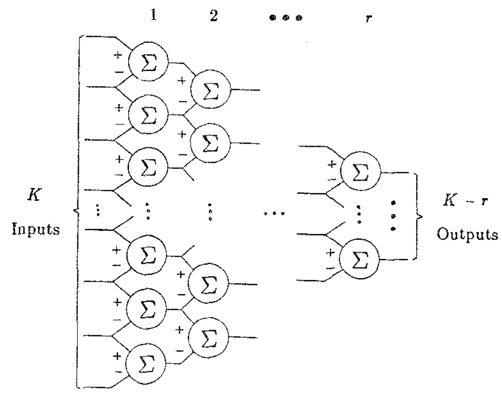

Fig. 4. Cascaded columns of differencing.

The simplest case is the $(K-1) \times K$ matrix $\mathbf{B}_K^{(0)}$, or $r = 1$, which is just adjacent element differencing, and configures the GSC to be ML.

In [2] it is shown that if in addition to all the previous assumptions, the array imperfections are "small," then the half-power adaptive beamwidth is

$$\boxed{HPBW \cong \frac{\lambda}{\pi d} \left(\frac{5\,(1 + SNR_i\,(K-1)\,\sigma_g^2)}{SNR_i\,K\,(K^2-1)} \right)^{1/2} \text{rad}} \quad (2)$$

where d is the ideal interelement spacing and σ_g^2 the ***gain error variance***, defined by $\sigma_g^2 \triangleq E_a\,[\,|\Delta g_{i,s}|^2\,]$ for all i, with the quantities $E_a\,[\cdot]$ and $\Delta g_{i,s}$ in turn defined as expectation over an ensemble of *i.i.d.* antenna elements, and the complex gain error at element i as seen by the signal, respectively.

Following Walach [3], a broadside look-direction was assumed. For look-directions θ_0 other than broadside, Walach pointed out how, using simple geometrical considerations, dividing by $\cos\theta_0$ approximates the *HPBW* formula, as long as θ_0 is small.

3. SIMULATIONS

Simulations will be described which verified (2).

The array used in the simulations was a 10-element line array with half-wavelength interelement spacing and a broadside look-direction, as shown in Fig. 5. The signal blocking matrix $\mathbf{B}_K^{(0)}$ was used to configure the GSC in a ML form. SNR_i was varied from -10 dB to 30 dB in 10 dB steps, and at each one of these steps random Gaussian element gain errors were generated, assumed to be equally due to amplitude and phase imperfections drawn from statistically independent zero-mean Gaussian distributions, so that $\sigma_g^2 \cong \sigma_a^2 + \sigma_p^2$, where $\sigma_a^2 \triangleq E_a[\,\Delta a_i^2\,]$ represents the *amplitude error variance*, and $\sigma_p^2 \triangleq E_a[\,\Delta p_i^2\,]$ the *phase error variance*. All values of Δg_i having a magnitude greater than $3\sigma_g$ were discarded, and new values regenerated in their place.

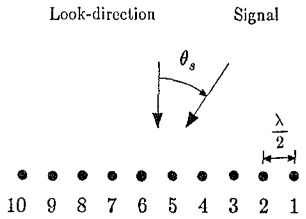

Fig. 5. Array geometry.

The beamformer was adapted to minimize *MSE* without being aware of the array imperfections corrupting the data, as previously assumed. For each trial, a curve of ***array gain*** γ_s **vs. *signal angle of arrival*** θ_s was obtained, and this curve was then expanded in the main beam region so that *HPBW* could be carefully measured using an interpolation routine. DOUBLE PRECISION COMPLEX was used on a DEC VAX 11/780.

For $SNR_i \geq 0$ dB, the agreement between the formula and the simulated data was good. For this range of SNR_i, the formula had a tendency to overestimate *HPBW* by roughly 4%, which could be partly or mostly due to small inaccuracies in the simulation (e.g., random number generator), but in any case is not a serious discrepancy. However, for $SNR_i = -10$ dB, the formula overestimated *HPBW* by a little over 25%, which should indicate caution on the part of potential users.

The reader is also urged to study the formula in light of Mayhan's [9] stressing that $\sigma_g^2 < -40$ dB (i.e., σ_g^2 (dB) $\triangleq 10 \log_{10} \sigma_g^2$) is usually considered very difficult to achieve in practice.

4. DISCUSSION OF IMPERFECTIONS

4.1. Degradation in Performance

In order to see superresolution graphically, the array gain of a 10-element line array with half-wavelength spacing, a broadside look-direction, ideal elements, and receiver noise 60 dB below the signal was plotted, as shown in Fig. 6. The adaptive beamformer was chosen to be the now-familiar GSC in a ML configuration. The superresolution concept is clearly demonstrated by the much narrower *HPBW* of the adaptive beamformer as compared to the conventional beamformer. In fact, the measured 0.0025^o *HPBW* of the adaptive beamformer as compared to 10.13^o for the conventional beamformer indicates a theoretical improvement in resolving ability of about 4000!

For the purpose of getting a rough idea of the sensitivity of superresolution to imperfections, all elements were subjected to a random two-dimensional misplacement having a standard deviation of

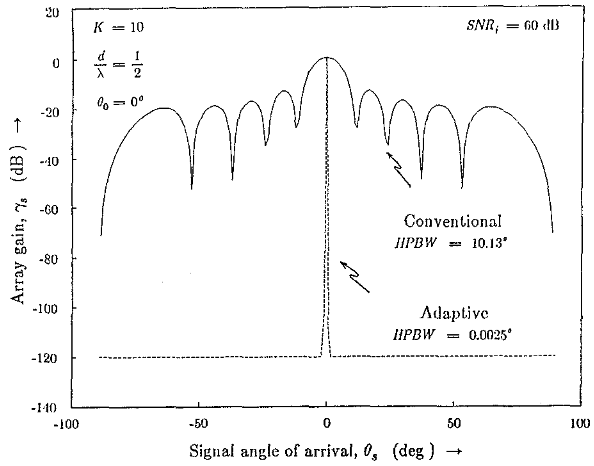

Fig. 6. Superresolution of array having ideal elements.

$\sigma_r = 0.03\,\lambda$ (6% of the ideal interelement spacing), as shown in Fig. 7. These random misplacements cause a direction-dependent phase error at each antenna element. In [4], it is shown that for far-field signals, under the three assumptions of Δx and Δy being "small," statistically independent, and equal, the phase error *variance* is independent of θ_s , and is given by $\sigma_p^2 \cong 4\pi^2(\sigma_r/\lambda)^2$.

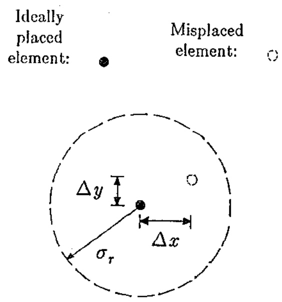

Fig. 7. Geometry of random element misplacement.

In this particular simulation, only phase errors were assumed, so $\sigma_g^2 = \sigma_p^2$, and is easily calculated to be -14.5 dB, a rather large error.

Once again using the weights which minimized *MSE* in the presence of the randomly generated imperfections, array gain of the imperfect array is plotted in Fig. 8, and the widening of the adaptive beamformer *HPBW* by a factor of about 320 displays its hypersensitivity to imperfections. Even with this terrible array, the resolving power of the adaptive beamformer is still over 10 times better than its conventional counterpart. Unfortunately, the superresolution may be very difficult to take advantage of, because array gain near the look-direction is so low.

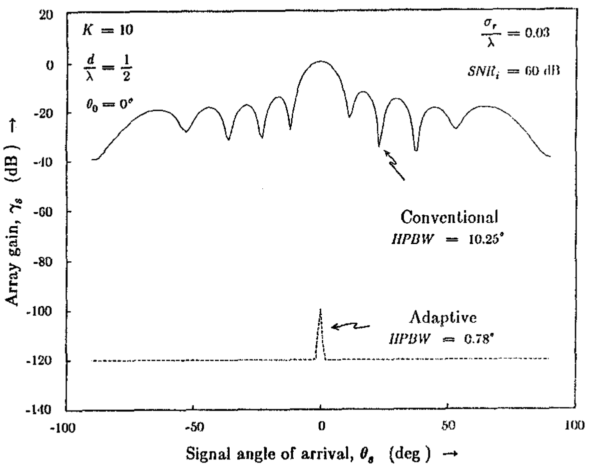

Fig. 8. Superresolution of array having misplaced elements.

4.2. A Simple and Effective Remedy

A technique for restoring the robustness of ML beamformers to imperfections is due to Zahm [6]. His idea was to *artificially* inject receiver noise in such a way that the weight vector was computed based on a higher receiver noise level than was actually present. However, the artificially injected noise does not actually appear at the beamformer output. This prevents the beamformer from nulling out signals very close to the look-direction.

Denoting the actual receiver noise power (as measured at any element) by σ_n^2 , and the artificially injected noise power by σ_i^2 , the resulting *equivalent* noise power σ_e^2 is just $\sigma_e^2 = \sigma_n^2 + \sigma_i^2$.

HPBW is only sensitive to the equivalent noise power. The same is not true of some other performance measures, such as *output signal-to-noise ratio* (*SNR_o*). A complete study of artificial noise injection in the GSC was performed by Jablon [4-5]. In the interest of brevity, only some of the main results will be summarized here, which consist of two possible on-line algorithms for artificially injecting receiver noise to make *HPBW* robust to element imperfections.

The first one is an extension of Widrow and Stearns' [7] *leaky LMS* algorithm to the GSC, and also turns out to be a special case of Chestek's [10] *soft-constrained LMS* algorithm. Before introducing the algorithm for artificial noise injection in the GSC, it is natural to first state the leaky LMS algorithm:

$$\mathbf{w}_{k+1} = (1 - \zeta)\,\mathbf{w}_k + 2\mu\,\bar{\epsilon}_k\,\mathbf{u}_k \tag{3}$$

where with reference to Fig. 1, $\mathbf{w}_k$ is the weight vector at time sample k, $(1-\zeta)$ the *leakage parameter*, μ the *adaptation constant*, ϵ_k the *error signal*, which for the GSC is chosen to be the beamformer output $z_k = d_k - y_k$, and $\mathbf{u}_k$ the state vector. The artificially injected receiver noise power then becomes $\sigma_i^2 = \zeta/(2\mu)$.

The *GSC leaky LMS* algorithm is [4-5]

$$\boxed{\mathbf{w}_{k+1} = (\mathbf{I} - \zeta\,\mathbf{B}\,\mathbf{B}^T)\,\mathbf{w}_k + 2\mu\,\bar{\epsilon}_k\,\mathbf{u}_k} \tag{4}$$

where ζ is a constant (generally close to zero) which satisfies $\zeta > 0$.

(4) has the effect of artificially injecting noise power of $\sigma_i^2 = \zeta/(2\mu)$. The signal power as measured at any element is σ_s^2 . Denoting the *input signal-to-equivalent-noise ratio* by $SER_i \triangleq \sigma_s^2/\sigma_e^2$, σ_i^2 is chosen in such a way that look-direction SNR_o is acceptable. In the look-direction, SNR_o may be approximated by [2]

$$SNR_o \cong \frac{SNR_i K}{(1 + SER_i(K-1)\sigma_g^2)^2 + SER_i^2 K(K-1)\sigma_g^2} \tag{5}$$

Chestek [10] guaranteed convergence of both the mean weight vector and *MSE* by choosing μ in accordance with

$$0 < \mu < \frac{1}{3\,Tr\,(\mathbf{R}_{uu}) + \dfrac{\zeta}{2\mu}\,Tr\,(\mathbf{B}\,\mathbf{B}^T)} \tag{6}$$

where $Tr\,(\cdot)$ denotes matrix *trace*, and $\mathbf{R}_{uu}$ autocovariance matrix. The bound (6) is easily calculable in an on-line implementation, since $Tr\,(\mathbf{R}_{uu})$ is just the total input power to the sidelobe cancelling branch, and ζ, μ, and $\mathbf{B}$ are all specified by the designer.

The advantage of using the LMS-type algorithm (4) over other adaptive schemes is that the computation per iteration required to update the weights is only on the order of $\hat{K}$ complex multiplications, since $\mathbf{B}\mathbf{B}^T$ is generally band-diagonal (e.g., $\mathbf{B}_{\hat{K}}^{(0)}(\mathbf{B}_{\hat{K}}^{(0)})^T$ is tridiagonal). Sometimes one wishes to estimate $\mathbf{R}_{uu}$ by other means, such as in the *sample matrix inversion* method considered by Reed *et al.* [11]. Then artificial noise can still be injected by

$$\boxed{\hat{\mathbf{R}}_{uu,i} = \hat{\mathbf{R}}_{uu} + \sigma_i^2\,\mathbf{B}\,\mathbf{B}^T} \tag{7}$$

where $\hat{\mathbf{R}}_{uu}$ is the data-dependent estimate of $\mathbf{R}_{uu}$, and $\hat{\mathbf{R}}_{uu,i}$ is the data-dependent estimate of $\mathbf{R}_{uu}$, modified by the artificial noise.

If artificial noise is injected into the ML GSC so that $SER_i = 10$ dB, corresponding to $SNR_o^* \cong 45$ dB, the array gain plot of Fig. 9 results (note change in scale). Although a comparison of Fig.'s 8 and 9 shows that the adaptive beamformer *HPBW* increased about 40% due to noise injection, there is no question that for practical applications, Fig. 9 is a great improvement over Fig. 8, since the array gain near the look-direction has been increased to almost the same value as for the conventional beamformer, while retaining the highly desirable low sidelobes. The resolving power of the adaptive beamformer is still about 10 times better than the conventional one. Further decreases in SNR_i (not shown) raised the sidelobes of the adaptive beamformer and both raised and widened its main lobe. By the time that SNR_i was -20 dB, the array gain plots of the adaptive and conventional beamformers were essentially the same, in agreement with the theory of artificial noise injection.

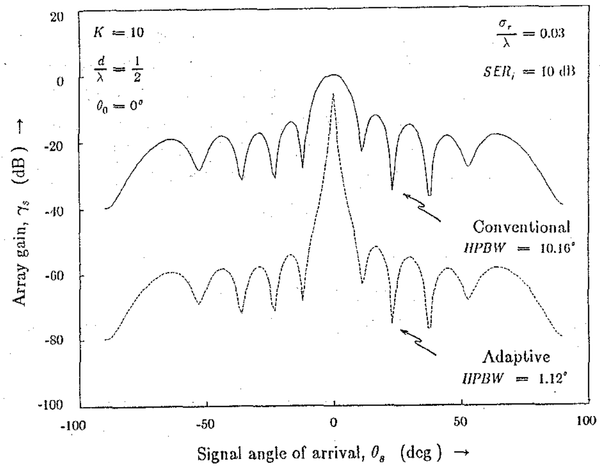

Fig. 9. Superresolution of array having misplaced elements with artificial noise added.

5. EXTENSION TO THE WIDEBAND CASE

The extension to the wideband case is trivial, assuming that *wideband* adaptive noise cancelling techniques [7] are now used, which give the optimal Wiener weightings as a function of frequency. The only differences in the wideband case are that one needs to keep track of the change in wavelength as a function of frequency (since it affects both presteering and any possible random element misplacement), and both SNR_i and σ_g^2 must be interpreted as functions of ω, which means that at some frequencies certain assumptions may no longer hold.

6. CONCLUSIONS

Array imperfections substantially degrade the *superresolution* capabilities of maximum-likelihood (ML) beamformers, as evidenced by the behavior of *half-power adaptive beamwidth*. However, these imperfections do not render the superresolution concept useless, since robustness to array imperfections can be improved tremendously by *artificially* injecting receiver noise.

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