cases may be summarized as follows:

Adaptive Filter	Fixed Filter	Conditions for Local Minima
$\frac{az}{z-b}$	$c_0+c_1z^{-1}$	$\left\lvert\frac{c_1}{c_0}\right\rvert>1$
$\frac{a_0z+a_1}{z-b}$	$c_0+c_1z^{-1}+c_2z^{-2}$	$\left\lvert\frac{c_2}{c_1}\right\rvert>1$
$\frac{az^2}{z^2-2b_1z-b_2}$	$c_0+c_1z^{-1}$	$\left\lvert\frac{c_1}{c_0}\right\rvert>1/2$

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REFERENCES

[1] K. J. Astrom and T. Soderstrom, "Uniqueness of the maximum likelihood estimates of the parameters of an ARMA model," *IEEE Trans. Automat. Contr.*, vol. AC-19, p. 769, Dec. 1974.
[2] C. R. Johnson, Jr., and M. G. Larimore, "Comments on and additions to 'An adaptive recursive LMS filter'," *Proc. IEEE* p. 1399, Sept. 1977.
[3] D. Parikh and N. Ahmed, "On an adaptive algorithm for IIR filters," *Proc. IEEE*, p. 585, May 1978.
[4] D. Soldan, N. Ahmed, and S. D. Stearns, "On using the sequential regression (SER) algorithm for long-term processing," in *Proc. 1980 IEEE ICASSP*, p. 1018, Apr. 1980.
[5] S. D. Stearns, "Error Surfaces of adaptive recursive filters," Sandia National Lab., SAND80-1348, Dec. 1980.
[6] J. R. Treichler, C. R. Johnson, and M. G. Larimore,"On the convergence properties of the simple hyperstable adaptive recursive filter," *Proc. 1980 IEEE ICASSP*, p. 997, Apr. 1980.
[7] B. Widrow *et al.*, "Adaptive noise cancelling: Principles and applications," *Proc. IEEE*, p. 1692, Dec. 1975.

Multichannel Adaptive Filtering for Signal Enhancement

EARL R. FERRARA, JR., AND BERNARD WIDROW

***Abstract*—An adaptive technique for enhancing a signal against additive noise is described. It makes use of two or more input channels containing correlated signal components but uncorrelated noise components. The various input signals need not be of the same waveshape, since the adaptive enhancer filters the inputs before summing them. The output is a best least squares estimate of the underlying signal in a chosen input channel. Adaptivity allows optimal performance even though the signal and noise characteristics differ from channel to channel and are unknown *a priori*.**

Formulas for signal distortion and output noise power are developed. The more input channels available containing correlated signal components, the better will be the system performance. Excellent performance is obtained when the sum of the filter input signal-to-noise ratios (SNR's), defined as functions of frequency, is large compared to unity at all frequencies of interest. In this case the output noise is small, the output signal distortion is small, and the output SNR is approximately equal to the sum of the filter input SNR's. As such, the multichannel adaptive signal enhancer is a generalization of the classic time-delay-and-sum beamforming antenna.

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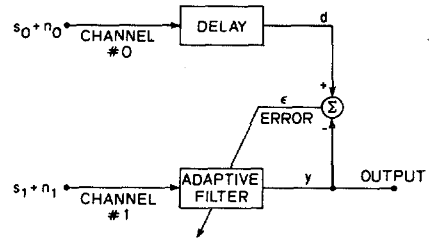

Fig. 1. A two-channel adaptive signal enhancer. The adaptive filter output is an estimate of s_0.

I. INTRODUCTION

A beamforming antenna is a simple example of a multichannel signal enhancer. In the conventional beamformer, an array of k antennas provides a set of k input channels which are delayed and summed to produce a useful output. Time delays are used to compensate for the various signal arrival times. Assuming that the receiver and antenna noises are uncorrelated from channel to channel (but of equal power) and that the signal components are identical after being aligned in time, adding the noisy signals yields an array output having a signal-to-noise ratio (SNR) which is improved by a factor of k over that of a single channel. Under these conditions, the SNR at the output is the sum of the input SNR's.

In many other practical situations the signal components among the available input channels may be related to each other in more complicated ways than mere time delay. In the cases of interest here, the various input signal components will differ in waveshape yet be correlated in unknown ways with each other. The noises will be mutually uncorrelated and uncorrelated with the signals. Their power spectra could differ from channel to channel.

Simply adding the channel inputs together will not suffice and could in fact be deleterious. Yet it would be advantageous to combine these inputs. We shall examine a method based on multichannel adaptive filtering for combining correlated noisy signals to enhance the output SNR. The resulting output noise and signal distortion will be investigated.

II. THE CONCEPT OF ADAPTIVE SIGNAL ENHANCING

Fig. 1 shows the block diagram of a two-channel adaptive signal enhancer. The zeroth-input channel contains signal s_0 plus noise n_0. Input channel number one contains a signal s_1, related to but not necessarily the same waveform as s_0, and an additive noise n_1. The noises n_0 and n_1 are assumed to be uncorrelated with each other and with both signals. The adaptive filter shown in Fig. 1 iteratively adjusts its impulse response via the LMS algorithm [1]–[5] (or via any other suitable algorithm) so that, after convergence, the power of the error ϵ, the difference between the filter output y and desired response d, is minimized. Ignoring the delay, the filter output is then a best least squares estimate of $d=s_0+n_0$. Since n_0 is uncorrelated with the filter input $x=s_1+n_1$, the filter output is a minimum mean-square error estimate of s_0 alone. A delay equal to half the adaptive filter

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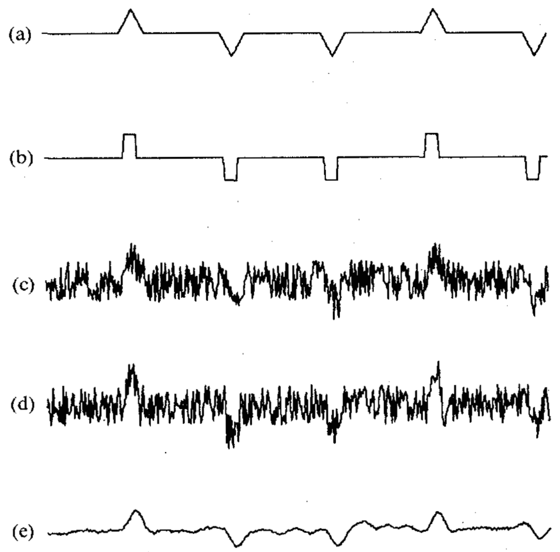

Fig. 2. Adaptive signal enhancing with a random pulse sequence. (a) Underlying waveform in desired response s_0. (b) Underlying waveform in filter input s_1. (c) Desired response, $d=s_0+n_0$. (d) Filter input, $x=s_1+n_1$. (e) Filter output—a best least squares estimate of (a).

length is included in the desired response of Fig. 1 in order to achieve the performance that would be obtained if the adaptive filter could be noncausal [4].

The most significant feature of adaptive signal enhancement is that the adaptive filter optimally estimates signal components without requiring explicit *a priori* knowledge of the statistical properties of the signal and noise components in either its input x or in its desired response d. What makes signal "signal" is that it is the mutually correlated part of x and d, while "noise" is the mutually uncorrelated part of x and d.

Fig. 2 shows pertinent waveforms for a two-channel signal enhancing experiment using computer simulated signals with a 41 weight adaptive transversal filter. The triangle pulse train of Fig. 2(a) is the underlying signal s_0, while the rectangle pulse train in Fig. 2(b) is the underlying signal s_1. The polarities of signal pulses in the two channels correspond, but vary randomly from pulse to pulse. Clearly the underlying signals are correlated. Uncorrelated white noises were added to the signals to form the two channel inputs which became the desired response and filter input, Fig. 2(c) and (d). The power of the noise in the filter input was twice that of the rectangle pulse train. The noise power in the desired response was three times that of the triangle pulse train. A pulse repetition interval of 80 s and a sampling frequency of 1 Hz was assumed. The adaptive signal enhancer output waveform, after convergence, is shown in Fig. 2(e). Noisy rectangular pulses were filtered to create an output which is a best least squares estimate of the triangle pulse train s_0.

III. Multichannel Adaptive Signal Enhancement

The idea of multichannel adaptive signal enhancement is illustrated in Fig. 3. The correlated input signal components are assumed to be generated from a common source s_{0j} by a convolutional process involving filters $G_1(z), G_2(z),\cdots,G_k(z)$. Uncorrelated noises are added. The resulting noisy inputs on channels 1–k are adaptively filtered and then summed to produce an output which is subtracted from the desired response (derived from channel #0) to produce an error signal ϵ_j. The weights of each adaptive filter are simultaneously adjusted via the LMS algorithm to minimize the power of the error ϵ_j. The output y_j is, after convergence of the adaptive filters, a best least squares estimate of the delayed input channel #0 (and of s_{0j} delayed since n_{0j} is uncorrelated with all other signals and noises).

Fig. 4 shows the underlying waveforms used in a multichannel

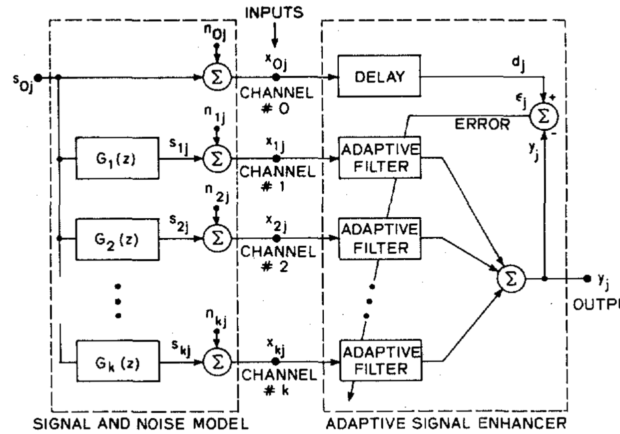

Fig. 3. Model for analysis of multichannel signal enhancer.

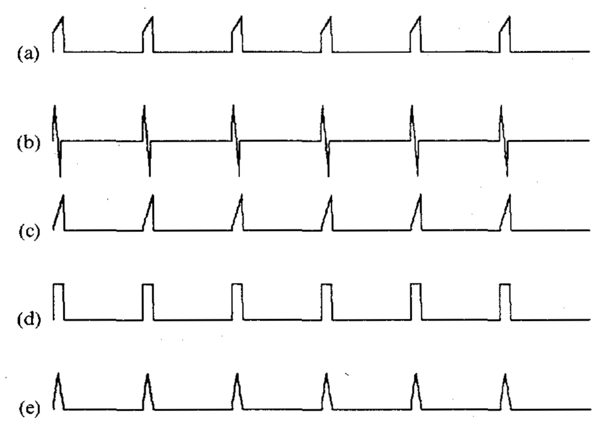

Fig. 4. Underlying waveforms for a multichannel adaptive signal enhancing simulation. (a) Underlying signal in channel 1. (b) Underlying signal in channel 2. (c) Underlying signal in channel 3. (d) Underlying signal in channel 4. (e) Underlying signal in desired response, channel 0.

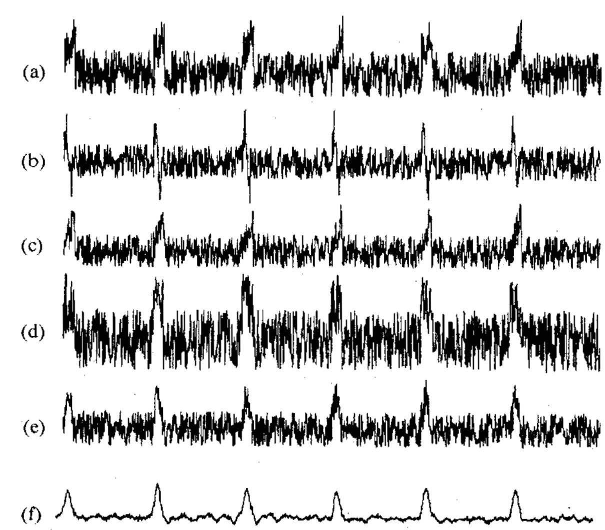

Fig. 5. Multichannel adaptive signal enhancing simulation. The output is a best least squares estimate of the triangle pulse train of Fig. 4(e). (a) Channel 1—filter input. (b) Channel 2—filter input. (c) Channel 3—filter input. (d) Channel 4—filter input. (e) Channel 0—desired response. (f) System output.

signal enhancing experiment. Uncorrelated white noises were added to these signals to obtain five noisy channel inputs, shown in Fig. 5, each having a total signal power-to-total noise power ratio of -3 dB. The channel containing the noisy triangle pulses was chosen as the desired response of the signal enhancer. The remaining four channels were used as filter inputs. Each filter had 30 weights. The enhanced output is shown in Fig. 5(f). Sharp triangles are clearly evident, having been manufactured from the noisy pulses of Fig. 5(a)–(d).

IV. Converged Solutions to Multichannel Adaptive Signal Enhancing Problems

In this section, optimal unconstrained Wiener solutions to a class of statistical signal enhancing problems are derived. For stationary stochastic inputs the steady-state performance (after convergence) of adaptive filters closely approximates that of Wiener filters, and Wiener filter theory thus provides a convenient method of mathematically analyzing adaptive signal enhancing problems. Though the idealized solutions presented do not take into account the issues of finite filter length or causality, means of approximating optimal unconstrained Wiener solutions (which may involve noncausal impulse responses[1] that could extend infinitely in both time directions) with adaptive transversal filters are available [4].

Consider a multi-input single-output Wiener filter, structured like the adaptive signal enhancer of Fig. 3. Let its inputs be $x_{1j},\cdots,x_{kj}$, its output y_j, and its desired response d_j. The inputs and output are assumed to be discrete in time, indexed by j, and the inputs and desired response are statistically stationary. The error signal is

$$\epsilon_j = d_j - y_j. \tag{1}$$

The filter is linear, discrete, and designed to be optimal in the minimum mean-square-error sense. In principle, it is composed of a set of infinitely long two-sided tapped delay lines. The tap weights are optimized.

The unconstrained Wiener solution is a set of optimal transfer functions, z-transforms of the tap weights, represented by the following vector [4]:

$$\mathcal{W}^*(z) \triangleq \begin{bmatrix} W_1^*(z) \\ W_2^*(z) \\ \vdots \\ W_k^*(z) \end{bmatrix} = \mathcal{R}^{-1}(z)\mathcal{P}(z). \tag{2}$$

The square matrix $\mathcal{R}(z)$ is the input spectral function defined by

$$\mathcal{R}(z) = \begin{bmatrix} \Phi_{x_1x_1}(z) & \Phi_{x_1x_2}(z) & \cdots & \\ \Phi_{x_2x_1}(z) & \Phi_{x_2x_2}(z) & & \\ \vdots & & \ddots & \\ & & & \Phi_{x_kx_k}(z) \end{bmatrix} \tag{3}$$

where $\Phi_{x_ix_j}(z)$ is the cross power density spectrum[2] between inputs x_i and x_j. Since the adaptive filter inputs contain signal plus uncorrelated noise, the power spectra are summed as follows:

$$\mathcal{R}(z) \triangleq \begin{bmatrix} \Phi_{n_1n_1}(z) & & \bigcirc \\ & \ddots & \\ \bigcirc & & \Phi_{n_kn_k}(z) \end{bmatrix} + \begin{bmatrix} \Phi_{s_1s_1}(z) & \Phi_{s_1s_2}(z) & \cdots & \\ \Phi_{s_2s_1}(z) & \Phi_{s_2s_2}(z) & \ddots & \\ \vdots & & & \Phi_{s_ks_k}(z) \end{bmatrix}$$
$$= \begin{bmatrix} \Phi_{n_1n_1}(z) & & \bigcirc \\ & \ddots & \\ \bigcirc & & \Phi_{n_kn_k}(z) \end{bmatrix} + \Phi_{s_0s_0}(z) \begin{bmatrix} \bar{G}_1(z)G_1(z) & \bar{G}_1(z)G_2(z) & \cdots & \\ \bar{G}_2(z)G_1(z) & \bar{G}_2(z)G_2(z) & & \\ \vdots & & \ddots & \\ & & & \bar{G}_k(z)G_k(z) \end{bmatrix}. \tag{4}$$

The bar over $G_1(z), G_2(z)$, etc., signifies complex conjugate. The first term in (4) is the input noise spectral function which will be denoted by $\Omega(z)$ and assumed positive definite:

$$\Omega(z) \triangleq \begin{bmatrix} \Phi_{n_1n_1}(z) & & \bigcirc \\ & \ddots & \\ \bigcirc & & \Phi_{n_kn_k}(z) \end{bmatrix}. \tag{5}$$

We next define a vector of transfer functions $\mathcal{G}(z)$ as

$$\mathcal{G}(z) = \begin{bmatrix} G_1(z) \\ G_2(z) \\ \vdots \\ G_k(z) \end{bmatrix}. \tag{6}$$

Now (4) becomes

$$\mathcal{R}(z) = \Omega(z) + \Phi_{s_0s_0}(z)\bar{\mathcal{G}}(z)\mathcal{G}^T(z). \tag{7}$$

The vector $\mathcal{P}(z)$ is the cross spectral function,

$$\mathcal{P}(z) = \begin{bmatrix} \Phi_{x_1d}(z) \\ \vdots \\ \Phi_{x_kd}(z) \end{bmatrix} = \begin{bmatrix} \Phi_{s_1s_0}(z) \\ \vdots \\ \Phi_{s_ks_0}(z) \end{bmatrix} = \Phi_{s_0s_0}(z)\bar{\mathcal{G}}(z). \tag{8}$$

Define the "SNR density function" at the input to the ith adaptive filter as

$$\mathrm{SNR}_i(z) \triangleq \frac{\Phi_{s_is_i}(z)}{\Phi_{n_in_i}(z)}. \tag{9}$$

This can be expressed as

$$\mathrm{SNR}_i(z) = \frac{G_i(z)\bar{G}_i(z)\Phi_{s_0s_0}(z)}{\Phi_{n_in_i}(z)}. \tag{10}$$

[1] To permit the causal adaptive filter to serve as a two-sided noncausal filter where required in practical situations, a delay is included as shown in Fig. 3. This delay is omitted in the analyses to follow. Instead, we allow the adaptive filter to be noncausal.

[2] We will express power density spectra as Z transforms with the understanding that they are only evaluated at real frequencies, i.e., for z on the unit circle.

The signal distortion waveform at the output of the converged multichannel enhancer can be calculated. Its z-transform is

$$\Delta(z)=S_0(z)-S_0(z)\mathcal{G}^T(z)\mathcal{W}^*(z). \quad (11)$$

Therefore, the power spectrum of the output signal distortion is

$$\Phi_{\Delta\Delta}(z)=\Phi_{s_0s_0}(z)|1-\mathcal{G}^T(z)\mathcal{W}^*(z)|^2. \quad (12)$$

Substituting (2), (7), and (8) in the right-hand term of (12) yields

$$1-\mathcal{G}^T(z)\mathcal{W}^*(z)=1-\mathcal{G}^T(z)\left[\bar{\mathcal{G}}(z)\mathcal{G}^T(z)+\frac{\Omega(z)}{\Phi_{s_0s_0}(z)}\right]^{-1}\bar{\mathcal{G}}(z). \quad (13)$$

After applying Lemma 1 of the Appendix we have

$$1-\mathcal{G}^T(z)\mathcal{W}^*(z)=1-\frac{\mathcal{G}^T(z)\Omega^{-1}(z)\bar{\mathcal{G}}(z)\Phi_{s_0s_0}(z)}{1+\mathcal{G}^T(z)\Omega^{-1}(z)\bar{\mathcal{G}}(z)\Phi_{s_0s_0}(z)}$$

$$=\left[1+\mathcal{G}^T(z)\Omega^{-1}(z)\bar{\mathcal{G}}(z)\Phi_{s_0s_0}(z)\right]^{-1}. \quad (14)$$

Recalling that $\Omega(z)$ is diagonal and using the definition of the signal-to-noise density function results in

$$1-\mathcal{G}^T(z)\mathcal{W}^*(z)=\left[1+\sum_{i=1}^{k}\mathrm{SNR}_i(z)\right]^{-1}$$

$$=[1+\mathrm{SNR}_{\mathrm{eff}}(z)]^{-1}. \quad (15)$$

We have defined an "effective" signal-to-noise spectral density as the sum of the signal-to-noise spectral densities of channel 1–k, i.e.,

$$\mathrm{SNR}_{\mathrm{eff}}(z)=\sum_{i=1}^{k}\mathrm{SNR}_i(z). \quad (16)$$

Substituting (15) in (12) yields the signal distortion spectrum as

$$\Phi_{\Delta\Delta}(z)=\Phi_{s_0s_0}(z)\frac{1}{[1+\mathrm{SNR}_{\mathrm{eff}}(z)]^2}. \quad (17)$$

The output noise power spectrum is

$$\Phi_{nn}(z)=\bar{\mathcal{W}}^{*T}(z)\Omega(z)\mathcal{W}^*(z) \quad (18)$$

which on substituting (2), (7), and (8) becomes

$$\Phi_{nn}(z)=\mathcal{G}^T(z)\left[\bar{\mathcal{G}}(z)\mathcal{G}^T(z)+\frac{\Omega(z)}{\Phi_{s_0s_0}(z)}\right]^{-1}$$

$$\cdot\Omega(z)\left[\bar{\mathcal{G}}(z)\mathcal{G}^T(z)+\frac{\Omega(z)}{\Phi_{s_0s_0}(z)}\right]^{-1}\bar{\mathcal{G}}(z). \quad (19)$$

Application of Lemma 2 of the Appendix to (19) yields

$$\Phi_{nn}(z)=\Phi_{s_0s_0}(z)\frac{\mathcal{G}^T(z)\Omega^{-1}(z)\bar{\mathcal{G}}(z)\Phi_{s_0s_0}(z)}{\left[1+\mathcal{G}^T(z)\Omega^{-1}(z)\bar{\mathcal{G}}(z)\Phi_{s_0s_0}(z)\right]^2}. \quad (20)$$

After using (9), (10), and (16), we have the output noise power spectrum as

$$\Phi_{nn}(z)=\frac{\mathrm{SNR}_{\mathrm{eff}}(z)}{[1+\mathrm{SNR}_{\mathrm{eff}}(z)]^2}\Phi_{s_0s_0}(z). \quad (21)$$

Since signal and noise are assumed uncorrelated, the spectrum of the output estimation error is the sum of that due to signal distortion and that due to noise. Summing (21) and (17), then dividing by $\Phi_{s_0s_0}(z)$, yields a *normalized* estimation error spectrum for the $k+1$ channel signal enhancer:

$$(\text{Normalized Estimation Error Spectrum})=\frac{1}{[1+\mathrm{SNR}_{\mathrm{eff}}(z)]}. \quad (22)$$

A special case of this result pertains to the two-channel enhancer of Fig. 1. Here,

$$(\text{Normalized Estimation Error Spectrum})=\frac{1}{[1+\mathrm{SNR}_1(z)]}. \quad (23)$$

It is clear that estimation error diminishes as SNR increases. It also diminishes as the number of channels increases.

Comparing (22) with (23) shows that the performance of the $k+1$ channel signal enhancer is identical to that of a two-channel enhancer whose filter input SNR density function is equal to the sum of the SNR density functions of the k individual filters.

For a given level of SNR on the available input channels, one would like to use a sufficient number of inputs so that, for all z on the unit circle,

$$\mathrm{SNR}_{\mathrm{eff}}(z)\gg 1. \quad (24)$$

When this condition is met, (17) shows that the signal distortion will be only a small fraction of the output signal, since

$$\frac{\Phi_{\Delta\Delta}(z)}{\Phi_{s_0s_0}(z)}=\frac{1}{[1+\mathrm{SNR}_{\mathrm{eff}}(z)]^2}\ll 1. \quad (25)$$

Therefore, the signal component at the system output will be essentially equal to s_{0j} and the ratio of the output signal spectrum to output noise spectrum is closely approximated by

$$\text{output SNR}(z)\triangleq\frac{\text{output signal spectrum}}{\text{output noise spectrum}}\approx\frac{\Phi_{s_0s_0}(z)}{\Phi_{nn}(z)}. \quad (26)$$

Making use of (21), we obtain

$$\text{output SNR}(z)\approx\frac{\Phi_{s_0s_0}(z)}{\Phi_{nn}(z)}=\frac{[1+\mathrm{SNR}_{\mathrm{eff}}(z)]^2}{\mathrm{SNR}_{\mathrm{eff}}(z)}\approx\mathrm{SNR}_{\mathrm{eff}}(z). \quad (27)$$

When condition (24) is met, the adaptive signal enhancer is being used properly. Its output SNR density function is then approximately equal to the sum of the SNR densities of the k individual filter inputs. The $k+1$ channel signal enhancer with a complicated and unknown signal structure behaves similarly to a conventional k element time-delay-and-sum beamformer with a simple known signal structure.

V. Conclusions

An adaptive technique for enhancing a signal against additive noise has been described. The method requires the availability of two or more input channels containing correlated signal components but uncorrelated noise components. The adaptive signal enhancer discriminates between signal and noise on the basis of their channel-to-channel correlation. Formulas for the output signal distortion power spectrum and the noise output power spectrum have been derived for the signal enhancer. They are functions of the frequency dependent SNR's at the inputs to the adaptive filters.

The effect of number of input channels on system performance has been analyzed. The $k+1$ channel enhancer was shown to be exactly equivalent in performance to a two channel enhancer whose adaptive filter input SNR is the sum of the SNR's of the k individual adaptive filter inputs. When this sum is much greater than one at all real frequencies, the adaptive multichannel enhancer is being used properly. Under this condition, the output signal distortion and the output noise level are low. The output SNR density is approximately equal to the sum of the SNR densities of the k individual adaptive filter inputs. Therefore, the behavior of the $k+1$ channel adaptive signal enhancer with a complicated and unknown signal and noise structure is similar to that of the k channel time-delay-and-sum beamforming array described above whose inputs are simple, containing equal signals of known relative time delay and noises of equal power.

Appendix

Two lemmas used in the analysis of the multichannel signal enhancer are proved below.

Lemma 1. Let Y be a complex vector and A be a nonsingular Hermitian matrix.

Then

$$\bar{Y}^T(A+Y\bar{Y}^T)^{-1}Y=\frac{\bar{Y}^TA^{-1}Y}{1+\bar{Y}^TA^{-1}Y}.$$

Proof: The following matrix inversion formula:

$$(A+Y\bar{Y}^T)^{-1}=A^{-1}-\frac{A^{-1}Y\bar{Y}^TA^{-1}}{1+\bar{Y}^TA^{-1}Y} \tag{A.1}$$

may be verified by multiplying both sides by $(A+Y\bar{Y}^T)$. Using (A.1) we have

$$\bar{Y}^T(A+Y\bar{Y}^T)^{-1}Y=\bar{Y}^TA^{-1}Y-\frac{(\bar{Y}^TA^{-1}Y)(\bar{Y}^TA^{-1}Y)}{1+\bar{Y}^TA^{-1}Y}$$
$$=\frac{\bar{Y}^TA^{-1}Y}{1+\bar{Y}^TA^{-1}Y}$$

proving the lemma.

Lemma 2. Let Y and A be as in Lemma 1. Then

$$\bar{Y}^T(A+Y\bar{Y}^T)^{-1}A(A+Y\bar{Y}^T)^{-1}Y=\frac{\bar{Y}^TA^{-1}Y}{(1+\bar{Y}^TA^{-1}Y)^2}.$$

Proof: Using (A.1) we have

$$\bar{Y}^T(Y+Y\bar{Y}^T)^{-1}A(A+Y\bar{Y}^T)^{-1}Y$$
$$=\bar{Y}^T\left(A^{-1}-\frac{A^{-1}Y\bar{Y}^TA^{-1}}{1+\bar{Y}^TA^{-1}Y}\right)A(A+Y\bar{Y}^T)^{-1}Y$$
$$=\bar{Y}^T\left(I-\frac{A^{-1}Y\bar{Y}^T}{1+\bar{Y}^TA^{-1}Y}\right)(A+Y\bar{Y}^T)^{-1}Y$$
$$=\left(1-\frac{\bar{Y}^TA^{-1}Y}{1+\bar{Y}^TA^{-1}Y}\right)\bar{Y}^T(A+Y\bar{Y}^T)^{-1}Y$$
$$=\frac{1}{1+\bar{Y}^TA^{-1}Y}\bar{Y}^T(A+Y\bar{Y}^T)^{-1}Y.$$

Application of Lemma 1 yields the desired result

$$\bar{Y}^T(A+Y\bar{Y}^T)^{-1}A(A+Y\bar{Y}^T)^{-1}Y=\frac{\bar{Y}^TA^{-1}Y}{(1+\bar{Y}^TA^{-1}Y)^2}.$$

References

[1] B. Widrow and M. E. Hoff, Jr., "Adaptive switching circuits," in *IRE WESCON Conv. Rec.*, pt. 4, pp. 96–104, 1960.

[2] B. Widrow, "Adaptive filters," in *Aspects of Network and System Theory*, R. Kalman and N. DeClaris, Eds. New York: Holt, Rinehart, and Winston, pp. 563–587, 1971.

[3] B. Widrow, P. Mantey, L. Griffiths, and B. Goode, "Adaptive antenna systems," *Proc. IEEE*, vol. 55, pp. 2143–2159, Dec. 1967.

[4] B. Widrow *et al.*, "Adaptive noise cancelling: Principles and applications," *Proc. IEEE*, vol. 63, pp. 1692–1716, Dec. 1975.

[5] B. Widrow *et al.*, "Stationary and nonstationary learning characteristics of the LMS adaptive filter," *Proc. IEEE*, vol. 64, pp. 1151–1162, Aug. 1976.

A Comparison of LMS Adaptive Cancellers Implemented in the Frequency Domain and the Time Domain

FRANCIS A. REED AND PAUL L. FEINTUCH

***Abstract*—Adaptive cancelling can be performed in the frequency domain with significant computational savings over time-domain implementations. This paper considers the statistical behavior of a frequency-domain adaptive canceller with white noise inputs, and develops expressions for the mean and variance of the adaptive filter weights, and for the mean-square error (MSE). These are compared to the behavior of a time-domain canceller with the same inputs through a combination of analysis and simulation. It is shown that the performance of the two algorithms can differ significantly due to the effects of block processing in the FFT. However, conditions are given under which the two implementations are essentially equivalent for white noise inputs so that the frequency-domain algorithm can be used to predict the mean, variance, time response, and MSE of the time-domain algorithm.**

I. Introduction

The LMS adaptive filter algorithm [1] has found many applications in situations where statistics of the input processes are unknown or changing. These include noise cancelling [2], line enhancing [2], [3], and adaptive array processing [4]. The algorithm utilizes a nonrecursive filter structure driven by a primary input. The filter weights are updated iteratively based upon the difference between the filter output and a reference input, so as to minimize the mean square value of this difference.

Implementation of this algorithm in the frequency domain [5] can give significant reductions in computation over the conventional time-domain approach. Under certain conditions, it is possible to compute both the mean and variance of the weights of the frequency-domain adaptive filter using [6], which is not generally possible in the time domain. The frequency-domain algorithm is not an exact realization of the time-domain version, however, and may yield different results.

This paper considers the behavior of a frequency-domain adaptive filter configured as a broad-band canceller with white Gaussian inputs. Expressions are developed for the transient and steady-state mean weights, the steady-state variance of the weights and the steady-state mean squared error. These are compared to the behavior of a time-domain canceller with the same inputs through a combination of analysis and simulation.

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